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1989 J. Phys. A: Math. Gen. 22 L461

(http://iopscience.iop.org/0305-4470/22/11/003)

J. Phys. A: Math. Gen. 22 (1989) L461-L466. Printed in the UK

LETTER TO THE EDITOR

Parallel transport along a space curve and related phases

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Received 27 February 1989

Abstract. We investigate geometrical properties of a space curve and of its spherical images. We show that a space curve is characterised by two 'phase-like' quantities and comment on the relation of these quantities to the Berry phase.

In recent years there has been increased interest in phenomena which are associated with topological properties of the parameter space of a system under study. In particular, in a series of interesting papers, Berry [1] showed that a quantum system, which depends on some parameters and which evolves in time in such a way that during the evolution the state of the system traces out a closed curve in the space of these parameters can, in addition to the 'usual' dynamical phase, pick up also a 'topological' phase. This latter phase is associated with the motion of the system in the space of parameters.

Although the overall phase of a quantum system is normally not observable, Berry [1] discussed the cases when the topological phase of a subsystem can be observed through the interference with the phase of the other subsystem.

A classic example of such a situation involves polarised light in waveguides, which are split and later, after one of the guides is acted upon by appropriate forces or just follows an appropriately twisted path, are recombined. The inteference of the recombined light provides us with information about the relative phases of light in both waveguides.

This classic example has been studied by many people. In particular Ross [2], Tomita and Chiao, and Chiao and Wu [3] discussed this phenomenon for various configurations of the thin waveguides (optical fibres). Moreover Ross [2] experimentaly verified this effect for the case of a helical form of the fibre. More recently Haldane [4] and Kugler and Shtrikman [5] stressed the geometrical nature of the effect and pointed out that the phenomenon of the phase dependence of the rotation of polarisation in optical fibres can be understood in terms of the parallel transport along the fibre of the unit vector characterising the curve described by the fibre.

In this letter we study in detail the geometrical properties of a space curve (in the physical context this curve represents the path traced out in the parameter space by the state of the system; in particular it could represent an optical fibre, etc). We concentrate our attention on the spherical images of the curve (the curves traced out

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0305-4470/89/110461+06$02.50

on S^2 by the tangent, normal and binormal vectors of the space curve). We find that, in general, a space curve is characterised by two 'phase-like' quantities: the integral over the torsion and the integral over the curvature both calculated along the curve.

As Kugler and Shtrikman [5] have pointed out, the first of these quantities, in the context of the waveguides, corresponds to the Berry phase; as far as we are aware, the second one has not yet been shown to be physically observable. However, from the purely geometrical point of view both quantities are equally important and as we show below are closely related.

A space curve is described either by its parameter equations or by its natural equations: $\kappa = \kappa(s)$, $\tau = \tau(s)$, where κ, τ and s are the curvature, the torsion and the length (treated as the natural parameter) of the space curve. Let us consider a curve γ which in its parametric form is described by $\boldsymbol{r} = \boldsymbol{r}(s)$. Let us also denote by $\boldsymbol{t}$ the unit tangent vector to this curve and by $\boldsymbol{n}$ and $\boldsymbol{b}$ its principal normal and binormal respectively. Then the three vectors $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$ form a moving triad of the curve.

As is well known, they are related by the Frenet–Serret formulae:

$$
\begin{aligned}
\dot{\boldsymbol{t}} &= \kappa \boldsymbol{n} \\
\dot{\boldsymbol{n}} &= -\kappa \boldsymbol{t} + \tau \boldsymbol{b} \\
\dot{\boldsymbol{b}} &= -\tau \boldsymbol{n}
\end{aligned}
\tag{1}
$$

where the overdot denotes $\mathrm{d}/\mathrm{d}s$. If we parallel transport these vectors to the origin of the Cartesian coordinate system then, as these vectors are all of unit lengths, as the triad moves along the original space curve their ends generate three curves on the unit sphere S^2. These curves on S^2 are called spherical images of the space curve γ or its tangent, principal normal and binormal indicatrices of the space curve.

The elements of length along these curves $\mathrm{d}s_t$, $\mathrm{d}s_n$ and $\mathrm{d}s_b$ are given by:

$$
\begin{aligned}
\mathrm{d}s_t^2 &= \kappa^2\, \mathrm{d}s^2 \\
\mathrm{d}s_n^2 &= (\kappa^2 + \tau^2)\mathrm{d}s = \mathrm{d}s_t^2 + \mathrm{d}s_b^2 \\
\mathrm{d}s_b^2 &= \tau^2\, \mathrm{d}s^2.
\end{aligned}
\tag{2}
$$

If we introduce the Darboux vector [6] $\boldsymbol{\xi}$ the Frenet–Serret equations (1) can be rewritten as

$$
\begin{aligned}
\dot{\boldsymbol{t}} &= \boldsymbol{\xi} \wedge \boldsymbol{t} \\
\dot{\boldsymbol{n}} &= \boldsymbol{\xi} \wedge \boldsymbol{n} \\
\dot{\boldsymbol{b}} &= \boldsymbol{\xi} \wedge \boldsymbol{b}
\end{aligned}
\tag{3}
$$

where $\boldsymbol{\xi} = \tau \boldsymbol{t} + \kappa \boldsymbol{b}$. Observe that when a point moves with a unit velocity along the space curve the angular velocity of the triad is given by $\boldsymbol{\xi}$. As $\boldsymbol{\xi}$ is *not* a derivative of any vector $\boldsymbol{r} = \boldsymbol{r}(s)$, it is sometimes called a non-holonomic vector. Therefore $\tau\, \mathrm{d}s$ and $\kappa\, \mathrm{d}s$ are closed but not exact 1-forms; closed contour integrals over these 1-forms will give non-zero contributions.

The angular velocity has components along the tangent vector $\boldsymbol{t}$ and along the binormal vector $\boldsymbol{b}$. Let us consider the plane perpendicular to the tangent $\boldsymbol{t}$, which moves along the curve γ with a constant unit velocity. The vectors $\boldsymbol{n}$ and $\boldsymbol{b}$ span this plane. However, the natural frame ($\boldsymbol{n}$ and $\boldsymbol{b}$) rotates around $\boldsymbol{t}$ with an angular velocity $\tau(s)$. Thus after s has increased from $s = 0$ to $s = s_0$ the system develops a phase $\phi_1 = \int_0^{s_0} \tau(s)\, \mathrm{d}s$ between $\boldsymbol{n}$, $\boldsymbol{b}$ and the corresponding non-rotating frame in this plane.

Such a non-rotating frame could be defined by using the usual Fermi–Walker parallel transport along the curve γ [7]

$$\frac{\mathrm{D}A^i}{\mathrm{d}s}=\kappa A^k(t^k n^i - t^i n^k)=\{\kappa \boldsymbol{b}\wedge \boldsymbol{A}\}^i. \qquad (4)$$

It is this phase that appears, for example, in the process of travelling of light along a twisted waveguide.

However, let us consider the tangent plane to the curve spanned by $\boldsymbol{n}(s)$ and $\boldsymbol{t}(s)$, which also travels with a constant velocity along the curve. The natural axes in this plane are now given by $(\boldsymbol{t}, \boldsymbol{n})$, which rotate around $\boldsymbol{b}$, as the plane moves along the curve, with an angular velocity $\kappa(s)$. If at $s=0$ the natural axes and the axes of a non-rotating frame coincide at $s=s_0$ they differ by a phase $\phi_2=\int_0^{s_0}\kappa(s)\,\mathrm{d}s$. Here we have assumed that our non-rotating frame can be defined in a way similar to the usual Fermi–Walker transport; namely by

$$\frac{\mathrm{D}A^i}{\mathrm{d}s}=\tau A^k(n^k b^i - n^i b^k)=\{\tau \boldsymbol{t}\wedge \boldsymbol{A}\}^i. \qquad (5)$$

However, in the present case, it is the binormal $\boldsymbol{b}$, and not the tangent $\boldsymbol{t}$ as in the usual Fermi–Walker parallel transport, that is parallel transported along the curve.

Let us finally mention that the usual Fermi–Walker parallel transport as well as our modified Fermi–Walker parallel transport are special cases of a Frenet–Serret parallel transport along the curve γ which is an obvious generalisation of (3), namely

$$\frac{\mathrm{D}A^i}{\mathrm{d}s}=\{\kappa \boldsymbol{b}\wedge \boldsymbol{A}+\tau \boldsymbol{t}\wedge \boldsymbol{A}\}^i=\{\boldsymbol{\xi}\wedge \boldsymbol{A}\}^i. \qquad (6)$$

The two phases are very similar in their geometric nature. To see this let us consider the Euclidean covariant derivative of a vector along the curve γ:

$$\nabla_t \boldsymbol{A}=\frac{\mathrm{d}x^i}{\mathrm{d}s}\left(\frac{\mathrm{d}A^k}{\mathrm{d}x^i}+A^j\Gamma^k_{ij}\right)\boldsymbol{e}_k \qquad (7)$$

where $\dot{\boldsymbol{\gamma}}=\boldsymbol{t}$, $\boldsymbol{e}_i=\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}$. Thus

$$\nabla_t A^k=\frac{\mathrm{d}A^k}{\mathrm{d}s}+\frac{\mathrm{d}x^i}{\mathrm{d}s}\Gamma^k_{ij}A^j=\frac{\mathrm{d}A^k}{\mathrm{d}s}+B^k_j A^j. \qquad (8)$$

As the Euclidean connection is a pure gauge, the matrix B has the following form

$$B^k_j=(R^{-1}\dot{R})^k_j \qquad (9)$$

where R is a rotation matrix, which can be parametrised by the Euler angles θ, ϕ and ψ. On the other hand $B^k_j=(\dot{e}_j)^k$, which is given by (1). Therefore $(\dot{e}_j)^k$ is given by $(\dot{e}_j)^k=(R^{-1}\dot{R})^k_j$. When written in components this last condition gives us the following equations:

$$\begin{aligned}\omega_t &= \sin\theta\sin\psi\,\mathrm{d}\phi+\cos\psi\,\mathrm{d}\theta=\kappa(s)\,\mathrm{d}s\\ \omega_n &= \sin\theta\cos\psi\,\mathrm{d}\phi-\sin\psi\,\mathrm{d}\theta=0 \qquad (10)\\ \omega_b &= \cos\theta\,\mathrm{d}\phi+\mathrm{d}\psi=\tau(s)\,\mathrm{d}s\end{aligned}$$

where θ, ϕ and ψ are functions of s, which in what follows we shall treat as 'time'.

We can now use the condition of the vanishing of ω_n and after some algebra we find that

$$\begin{aligned}\kappa\,\mathrm{d}s &= \sqrt{(\mathrm{d}\theta)^2+\sin^2\theta(\mathrm{d}\phi)^2}\\ \tau\,\mathrm{d}s &= \cos\theta\,\mathrm{d}\phi+\mathrm{d}\psi.\end{aligned} \tag{11}$$

Notice that if we restrict ourselves to curves $\gamma(s)$ such that $\boldsymbol{t}(0)$, $\boldsymbol{n}(0)$ and $\boldsymbol{b}(0)$ are parallel, respectively, to $\boldsymbol{t}(s_0)$, $\boldsymbol{n}(s_0)$ and $\boldsymbol{b}(s_0)$ then the spherical indicatrices of these curves correspond to closed curves on S^2.

Next we consider the integrals of the curvature and the torsion of our space curve, i.e. the integrals

$$I_\kappa = \int_0^{s_0} \kappa(s)\,\mathrm{d}s \qquad \text{and} \qquad I_\tau = \int_0^{s_0} \tau(s)\,\mathrm{d}s. \tag{12}$$

From (2) we see that $I_\kappa = L_t$ (the length of the tangent indicatrix) and that $I_\tau = L_b$ (the length of the binormal indicatrix). However, using (11) and the Stokes' theorem we also have that

$$I_\tau = \oint \cos\theta\,\mathrm{d}\phi = 2\pi - \iint \sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi \tag{13}$$

where the last integral denotes the area bounded by the tangent indicatrix which we will denote by Ω_t. Observe that for a closed curve $\oint \mathrm{d}\psi = 0$ as using $\omega_n = 0$ we can show that $\mathrm{d}\psi(\theta,\phi)$ is an exact 1-form defined over the whole S^2. We see that, up to the additive 2π term, I_b and Ω_t are equal.

In fact a similar relation holds between L_t and Ω_b, where Ω_b denotes the area bounded by the binormal indicatrix. To show this we have to consider separately two different cases, namely: the right-handed ($\tau>0$) and the left-handed ($\tau<0$) curves. In the case of right-handed curves, let us consider the triad ($\boldsymbol{b}$, $-\boldsymbol{n}$, $\boldsymbol{t}$), where now $\boldsymbol{b}$ is the unit tangent vector to the new curve $\gamma_1(s)$ and $-\boldsymbol{n}$ and $\boldsymbol{t}$ are its principal normal and binormals respectively. The Frenet-Serret formulae for the curve $\gamma_1(s)$ now take the form

$$\begin{aligned}\dot{\boldsymbol{b}} &= \tau\boldsymbol{n}\\ \dot{\boldsymbol{n}} &= -\tau\boldsymbol{b}+\kappa\boldsymbol{t}\\ \dot{\boldsymbol{t}} &= -\kappa\boldsymbol{n}.\end{aligned} \tag{14}$$

The new curve $\gamma_1(s)$ is again a right-handed curve with curvature $\tau(s)$ and with torsion $\kappa(s)$. For the curve $\gamma_1(s)$, the length of its binormal L_t is equal (again, up to the addition of 2π) to the area Ω_b bounded by its tangent.

In the case of a left-handed curve $\tau<0$ and so we can introduce $\tau_0=-\tau$. Then we can consider the triad ($-\boldsymbol{b}$, $-\boldsymbol{n}$, $-\boldsymbol{t}$), for which $-\boldsymbol{b}$ is now the unit tangent vector, $-\boldsymbol{n}$ is its principal normal and $-\boldsymbol{t}$ is its binormal. The Frenet–Serret formulae are now

$$\begin{aligned}\dot{\boldsymbol{b}} &= \tau_0\boldsymbol{n}\\ \dot{\boldsymbol{n}} &= -\tau_0\boldsymbol{b}-\kappa\boldsymbol{t}\\ \dot{\boldsymbol{t}} &= \kappa\boldsymbol{n}.\end{aligned} \tag{15}$$

These equations describe now a new curve $\gamma_2(s)$, which is again left-handed and has curvature $\tau_0(s)$ and torsion $-\kappa(s)$.

As all spherical curves are related (they have related torsions and curvatures) we can relate their properties. Thus we have proved that, up to the addition of an overall 2π, $L_b = \Omega_t$ and $L_t = \Omega_b$.

Let us finally consider the principal normal indicatrix. Its line element is given (2) by

$$\begin{aligned} ds_n^2 &= ds_t^2 + ds_b^2 \\ &= (d\theta)^2 + (d\phi)^2 + (d\psi)^2 + 2\cos\theta\, d\phi\, d\psi \end{aligned} \tag{16}$$

where we have used (11). Equation (16) identifies ds_n with the line element of a curve on S^3 (of radius $R = 2$). The movement of the Frenet-Serret triad along the curve γ traces a curve on SO(3), parametrised by the Euler angles $\theta(s)$, $\phi(s)$ and $\psi(s)$. A Hopf map from S^3 onto S^2 would map this curve onto the tangent indicatrix with its line element ds_t.

The discussion given here can be generalised to higher-dimensional spaces. In particular, if we consider the case of four dimensions, we have to consider general tetrads and study rotations which connect two such general tetrads. In analogy with the three-dimensional case we can introduce generalised Euler angles and show that a general rotation can be resolved into six basic rotations in different 2-planes. A general curve in four dimensions is now characterised by three quantities: curvature, torsion and bitorsion, which all appear in the generalised Frenet-Serret formulae.

If we now impose the conditions on the general rotation matrix which follow from the general form of the Frenet-Serret formulae, we find (after some tedious algebra) that the integral over the bitorsion is again given by the length of some curve, except that this time the corresponding curve lies on S^3. This should be compared with the result in three dimensions where the corresponding quantity involves the torsion and the curve lies on S^2. It should be possible, in analogy with the three-dimensional case, to relate the length of this curve on S^3 to an integral over the area bounded by a related curve which has some topological meaning, but unfortunately, so far, we have not succeeded in this task.

Our discussion has shown that in three dimensions general curves are characterised by two local quantities, the curvature and torsion, and that they naturally lead to two phase-like quantities, one of which corresponds to the Berry phase. So far our discussion has been purely geometrical in nature. To see whether the additional phase we have discussed can be observed experimentally depends on the experimental set-up and its dynamics. In particular, in the dynamical systems in which one measures the phase of the polarised light beam in bent waveguides, the Berry phase arises from the interaction of the light beams with the waveguides which keep the polarisation of the wave transverse to the direction of propagation. Clearly, we would need to involve similar interactions but this time it is the normal component which should vanish or be altered as a result of the interaction. We have no specific proposal which exhibits such interactions and we leave it as a challenge to the reader to come up with an example of an appropriate experimental set-up to exhibit the more general phase. The obvious suggestions involve massive spinning particles which could include vector mesons (like W or Z) or even spin-$\frac{1}{2}$ particles like the neutron or proton. However, all massive vector mesons are unstable and it is hard to think of situations when the spin of the spin-$\frac{1}{2}$ particles rotates appropriately. But maybe one should seek applications elsewhere.

Of course the issue of whether these quantities (and in particular the second one) become phases in quantum systems and are observable is a separate and a non-trivial problem. This problem is currently under investigation and we hope to report on it in the near future.

WJZ would like to thank D K Campbell for useful conversations. RD would like to thank A R Bishop, D K Campbell and R Percacci for conversations and D K Campbell for hospitality during his stay at CNLS.

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